UNIT – V

PROBABILITY DISTRIBUTION – 1

RANDOM VARIABLE


BINOMIAL DISTRIBUTION

5.3. Definition

\[ P(X = x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2, \ldots, n \\ 0, & \text{Otherwise} \end{cases} \]

(Statement only) Expressions for mean and variance, Simple Problems

5.1 RANDOM VARIABLE

INTRODUCTION

Let a coin be tossed. Nobody knows what we will get whether a head or tail. But it is certain that either a head or tail will occur. In a similar way, if a dice is thrown, we may get any of the faces 1, 2, 3, 4, 5, and 6. But nobody knows which one will occur. Experiments of this type where the outcome cannot be predicted are called 'random' experiments.

The word probability or chance is used commonly in day –to–day life. For example the chances of India and South Africa winning the world cup cricket, before the start of the game are equal (i.e., 50:50). We often say that it will rain tomorrow. Probably I will not come to function today. All these terms – chance, probable, etc., convey the same meaning i.e., that event is not certain to take place. In other
words, there is an uncertainty about the happening of the event. The term probability refers to the randomness and uncertainty.

**TRAIL AND EVENT**

Consider an experiment of throwing a coin. When tossing a coin, we may get a head (H) or tail (T). Here tossing of a coin is a trail and getting a head or tail is an event.

From a pack of cards, drawing any three cards is trail and getting a king or a queen or a jack are events.

Throwing of a dice is a trail and getting 1 or 2 or 3 or 4 or 5 or 6 is an event.

**Sample space:**

The set of all possible cases of an experiment is called the sample space and is denoted by S

**Mathematical Definition of Probability**

Probability of happening an event \( E \)

\[
= \frac{\text{Number of favourable cases of the event}}{\text{Total number of exhaustive cases}}
\]

\[
= \frac{m}{n}
\]

Where \( m = \) number of favorable cases = \( n(E) \)

\( n = \) number of exhaustive cases = \( n(S) \)

**Random Variable:**

A function \( x \) which transforms events of a random experiment into real numbers is called random variable. It is denoted as \( X: S \rightarrow R \) where \( S \) is sample space of random experiment and \( R \) is set of real numbers

**Example:**

Two coins are tossed at a time

Sample space is \( S = \{HH, HT, TH, TT\} \)
If we take $X$ is the number of heads appearing then HH becomes 2, HT and TH becomes 1 and TT becomes 0

$\therefore X$ (number of heads) is a random variable

**TYPES OF RANDOM VARIABLES**

The are two types of random variables known as

(i) Discrete random variable

(ii) Continuous random variable

**Discrete random variable**

If a random variable takes only a finite or a countable number of values, it is called a discrete random variable.

For example, when two coins are tossed the number of heads obtained is the random variable $X$. $X$ assumes the values 0, 1, 2 which is a countable set. Such a variable is called discrete random variable.

**Definition: probability Mass Function:**

Let $X$ be a discrete random variable with values $x_1, x_2, x_3, \ldots, x_n$. Let $p(x_i)$ be a number associated with each $x_i$

Then the function $p$ is called the probability function of $X$ if it satisfies the conditions:

(i) $p(x_i) \geq 0$ for $i=1,2,3,\ldots,n$

(ii) $\sum p(x_i) = 1$

The set of ordered pairs $(x_i, p(x_i))$ is called the probability distribution of $X$. The probability function is also known as Probability Mass Function of $X$.

**Continuous Random Variable:**

A random variable $X$ is said to be continuous if it can take all possible values between certain limits.

**Examples:**

1. Life time of electric bulb in hours
2. Height, weight, temperature, etc.,
Definition: Probability density function:
A function $f$ is said to be probability density function (pdf) of the continuous random variable $X$ if it satisfies the following conditions:
1. $f(x) \geq 0$ for all $x \in \mathbb{R}$;
2. $\int_{-\infty}^{\infty} f(x) \, dx = 1$.

Definition:
Distribution function (Cumulative Distribution Function):
The function $F(x)$ is said to be the distribution function of the random variable $X$ if $F(x) = P(X \leq x); -\infty \leq x \leq \infty$.
The distribution function $F$ is also called Cumulative distribution function.

Note:
1. If $X$ is a discrete random variable then from the definition it follows that $F(x) = \sum p(x_i)$ where the summation is overall $X_i$ such that $x_i \leq x$.
2. If $X$ is a continuous random variable, then from the definition it follows that
$$F(x) = \int_{-\infty}^{x} f(t) \, dt \quad -\infty \leq x \leq \infty.$$ where $f(t)$ is the value of the probability density function of $X$ at $t$.

WORKED EXAMPLES
PART - A

1. Find the probability distribution of $X$ when tossing a coin, when $X$ is defined as setting a head.

Solution:
Let $X$ denote getting a head.
Probability of getting a head $= \frac{1}{2}$

Probability of getting a tail $= \frac{1}{2}$

$\therefore$ The probability distribution of $X$ is given by

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>
2. In a class of 10 students, 4 are boys and the rest are girls. Find the probability that a student selected will be a girl.

**Solution:**
Total number of students = 10  
Number of boys = 4  
Number of girls = 6

\[ \text{Probability that a girl is selected} = \frac{m}{n} = \frac{6}{10} = \frac{3}{5} \]

3. When throwing a die what is probability of getting a 4?

**Solution:**
Total number of cases \( n = 6 \)  
\{1,2,3,4,5,6\}  
Number of favorable cases = 1

\[ \text{Probability of getting} 4 = \frac{m}{n} = \frac{1}{6} \]

4. Find the chance that if a card is drawn at random from an ordinary pack, it is one of the court cards.

**Solution:**
Total no. of exhaustive cases = 52 cards.  
Number of favorable cases = 12  
(Court cards mean kings, queens, jacks. There are 4 × 3 = 12 court cards)

\[ \text{Probability} = \frac{12}{52} = \frac{3}{13} \]

5. A bag contains 7 white and 9 red balls. Find the probability of drawing a white ball.

**Solution:**
Number of favorable cases = 7  
Number of exhaustive cases = 16

\[ \text{Probability} = \frac{7}{16} \]

(One white ball can be drawn out of 7 in \( 7c_1 \) ways = 7 ways)
6. Verify that \( f(x) = \begin{cases} \frac{2x}{9}, & 0 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases} \) is a probability density function.

**Solution:**

\[
\int_{-\infty}^{\infty} f(x) \, dx = \int_{0}^{3} \frac{2x}{9} \, dx = \frac{2}{9} \left[ \frac{x^2}{2} \right]_{0}^{3} = \frac{2}{9} \cdot \frac{9}{2} = 1
\]

\[\Rightarrow f(x) \text{ is a probability density function}\]

7. A continuous random variable \( X \) has the pdf defined by

\[ f(x) = \begin{cases} ce^{-ax}, & 0 < x < \infty \\ 0, & \text{elsewhere} \end{cases} \]

Find the value of \( c \) if \( a > 0 \).

**Solution:**

\[
\int_{-\infty}^{\infty} f(x) \, dx = c \int_{0}^{\infty} e^{-ax} \, dx = c \left[ \frac{e^{-ax}}{-a} \right]_{0}^{\infty} = c \left[ 0 - \frac{1}{-a} \right] = \frac{c}{a}
\]

Since pdf is given \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

\[
\frac{c}{a} = 1 \\
\Rightarrow c = a
\]

**PART - B**

1. Find the probability mass function, and the cumulative distribution function for getting ‘3’ when two dice are thrown.

**Solutions:**

Let \( X \) is the random variable of setting number of ‘3’s. Therefore \( X \) can take the values 0,1,2.
Two dice are thrown, therefore total number of exhaustive cases is 36.

\[ S = \{(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
\]
\[ (2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
\]
\[ (3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
\]
\[ (4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
\]
\[ (5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
\]
\[ (6,1) (6,2) (6,3) (6,4) (6,5) (6,6)\}\]

Probability of no '3' = \( P(X = 0) = \frac{25}{36} \)

Probability of one '3' = \( P(X = 1) = \frac{10}{36} \)

Probability of two '3' = \( P(X = 2) = \frac{1}{36} \)

Probability mass function is given by

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>( \frac{25}{36} )</td>
<td>( \frac{10}{36} )</td>
<td>( \frac{1}{36} )</td>
</tr>
</tbody>
</table>

Cumulative distribution function

\[ F(x) = \sum_{x=-\infty}^{x} P(X = x) \]

\[ F(0) = P(X = 0) = \frac{25}{36} \]

\[ F(1) = P(X = 0) + P(X = 1) = \frac{25}{36} + \frac{10}{36} = \frac{35}{36} \]

\[ F(2) = P(X = 0) + P(X = 1) + P(X = 2) = \frac{25}{36} + \frac{10}{36} + \frac{1}{36} = \frac{36}{36} = 1 \]

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(x) )</td>
<td>( \frac{25}{36} )</td>
<td>( \frac{35}{36} )</td>
<td>1</td>
</tr>
</tbody>
</table>
2. A random variable $X$ has the following probability mass function

\[
\begin{array}{cccccccc}
  x & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
p(x) & k & 3k & 5k & 7k & 9k & 11k & 13k \\
\end{array}
\]

Find (i) $k$ (ii) Evaluate $P(X < 4)$, $P(X \geq 5)$, and $P(3 < x \leq 6)$

(iii) What is the smallest value of $x$ for which $P(X \leq x) > \frac{1}{2}$

**Solutions:**

i. Since $P(X=x)$ is the probability mass function, 

$$\sum_{i=0}^{6} p_i = 1$$

i.e., $k+3k+7k+9k+11k+13k=1$

$$49k = 1$$

$$k = \frac{1}{49}$$

ii $P(X < 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

\[= k + 3k + 5k + 7k\]

\[= 16k = \frac{16}{49}\]

\[P(X \geq 5) = P(X = 5) + P(X = 6) = 11k + 13k = 24k = \frac{24}{49}\]

\[P(3 < x \leq 6) = P(X = 4) + P(X = 5) + P(X = 6)\]

\[= 9k + 11k + 13k = 33k = \frac{33}{49}\]

iii The minimum value of $x$ may be determined by trial and error method.
\[ P(X \leq 0) = k = \frac{1}{49} < \frac{1}{2} \]

\[ P(X \leq 1) = k + 3k = 4k = \frac{4}{49} < \frac{1}{2} \]

\[ P(X \leq 2) = k + 3k + 5k = 9k = \frac{9}{49} < \frac{1}{2} \]

\[ P(X \leq 3) = k + 3k + 5k + 7k = 16k = \frac{16}{49} < \frac{1}{2} \]

\[ P(X \leq 4) = k + 3k + 5k + 7k + 9k = 25k = \frac{25}{49} > \frac{1}{2} \]

\[ \therefore \text{The smallest value of } x \text{ for which } P(X \leq x) > \frac{1}{2} \text{ is } 4 \]

3. A continuous random variable \( X \) has pdf

\[ f(x) = \begin{cases} 
  kx(1 - x)^{10}, & 0 < x < 1 \\
  0, & \text{otherwise}
\end{cases} \]

Find \( k \).

**Solution:**

Since \( f(x) \) is a pdf, we have \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

\[ \int_{0}^{1} kx(1 - x)^{10} \, dx = 1 \]

Put \( t = 1 - x \Rightarrow x = 1 - t \)

\[ dt = -dx \]

when \( x = 0, t = 1 \)

when \( x = 1, t = 0 \)
\[
\begin{align*}
&\left. k \int_0^1 (1-t) t^{10} (-dt) = 1 \right. \\
&\left. -k \left[ \frac{t^{11}}{11} - \frac{t^{12}}{12} \right]_1^0 = 1 \right. \\
&\left. -k \left[ 0 - \left( \frac{1}{11} - \frac{1}{12} \right) \right] = 1 \right. \\
&\left. k \left( \frac{12 - 11}{132} \right) = 1 \right. \\
&k = 132
\end{align*}
\]

4. For the pdf \( f(x) = \begin{cases} cx (1-x)^3 & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases} \)

Find

(i) the constant c.

(ii) \( P \left( X < \frac{1}{2} \right) \)

Solution:

(i) Since \( f(x) \) is a pdf, we have \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)

\[
\begin{align*}
\int_0^1 cx(1-x)^3 \, dx &= 1 \\
\int_0^1 c(1-x)(1-(1-x))^3 \, dx &= 1 \\
\int_0^1 c(1-x)x^3 \, dx &= 1 \\
\int_0^1 \frac{x^4}{4} - \frac{x^5}{5} \, dx &= 1 \\
\int_0^1 \left( \frac{1}{4} - \frac{1}{5} \right) = 1 \Rightarrow c = 20
\end{align*}
\]
\[
P(X < \frac{1}{2}) = \int_{0}^{\frac{1}{2}} 20x(1-x)^3 \, dx
\]

\[
= 20 \left[ x(1 - 3x + 3x^2 - x^3) \right]_{0}^{\frac{1}{2}}
\]

\[
= 20 \int_{0}^{\frac{1}{2}} (x - 3x^2 + 3x^3 - x^4) \, dx
\]

\[
= 20 \left\{ \frac{x^2}{2} - \frac{3x^3}{3} + \frac{3x^4}{4} - \frac{x^5}{5} \right\}^{1/2}_{0}
\]

\[
= 20 \left[ \frac{1}{8} - \frac{4}{8} + \frac{3}{4} \cdot \frac{1}{16} - \frac{1}{5} \cdot \frac{1}{32} \right]
\]

\[
= 20 \left[ \frac{3}{64} - \frac{1}{160} \right] = 20 \left[ \frac{15 - 2}{320} \right] = \frac{13}{16}
\]

5.2 MATHEMATICAL EXPECTATION OF DISCRETE VARIABLE

Expectation of a discrete random variable.

Definition: If \( X \) denotes a discrete random variable which can assume the value \( x_1, x_2, \ldots, x_n \) with respective probabilities \( p_1, p_2, \ldots, p_n \) then the mathematical expectation of \( X \), denoted by \( E(X) \) is defined by

\[
E(X) = p_1x_1 + p_2x_2 + \ldots + p_nx_n
\]

\[
= \sum_{i=1}^{n} p_i x_i \quad \text{where} \quad \sum_{i=1}^{n} p_i = 1
\]

Thus \( E(X) \) is the weighted arithmetic mean of the values \( x_i \) with the weight to \( p(x_i) \)

\[
\therefore \text{mean } \overline{X} = E(X)
\]

Hence the mathematical expectation \( E(X) \) of a random variable is simply the arithmetic mean.

Result: If \( \phi(x) \) is a function of a random variable \( X \), then

\[
E [\phi(x)] = \sum P(X=x) \phi(x).
\]
**Properties of mathematical expectation:**

1. \( E(c) = c \) where \( c \) is a constant.
2. \( E(cX) = cE(X) \) where \( c \) is a constant.
3. \( E(aX + b) = aE(X) + b \) where \( a \) & \( b \) are constants.
4. Variance of \( X = \text{var}(X) = E(X - E(X))^2 \)
5. \( \text{Var}(X) = E(X^2) - [E(X)]^2 \)
6. \( \text{Var}(X \pm c) = \text{Var}(X) \) where \( c \) is a constant.
7. \( \text{Var}(aX) = a^2 \text{Var}(X) \)
8. \( \text{Var}(aX + b) = a^2 \text{Var}(X) \)
9. \( \text{Var}(c) = 0 \) where \( c \) is a constant.

**WORKED EXAMPLE**

**PART - A**

1. Find the expected value of the number on a die when thrown.

   **Solution:**

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(x)</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
<td>1/6</td>
</tr>
</tbody>
</table>

   \[
   E(x) = \sum x_i P(x_i) = 1 \left( \frac{1}{6} \right) + 2 \left( \frac{1}{6} \right) + 3 \left( \frac{1}{6} \right) + 4 \left( \frac{1}{6} \right) + 5 \left( \frac{1}{6} \right) + 6 \left( \frac{1}{6} \right) = \frac{21}{6} = \frac{7}{2}.
   \]

2. Evaluate \( \text{Var}(2X \pm 3) \)

   We have, \( \text{Var}(aX \pm b) = a^2 \text{Var}(X) \)
   \[
   \text{Var}(2X \pm 3) = 2^2 \text{Var}(X) = 4 \text{Var}(X)
   \]
(3) A random variable \( X \) has \( \text{E}(X) = 2 \) and \( \text{E}(X^2) = 8 \). Find its variance.

\[
\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2 = 8 - 2^2 = 8 - 4 = 4.
\]

PART - B

(4) An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws when a ball is drawn at random with replacement. Also find its mean and variance.

Solution:

Let \( X \) be the random variable of drawing number of red balls in three draws.

\[
\therefore \text{X can take the values} \quad 0,1,2,3.
\]

\[
P(\text{Red ball}) = \frac{3}{7}; \quad P(\text{not a red ball}) = \frac{4}{7}
\]

\[
P(X = 0) = P(\text{WWW}) = \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{4}{7} = \frac{64}{343}
\]

\[
P(X = 1) = 3P(\text{RWW})
\]

\[
= 3 \cdot \frac{3}{7} \cdot \frac{4}{7} \cdot \frac{4}{7} = \frac{144}{343}
\]

\[
P(X = 2) = 3P(\text{RRW})
\]

\[
= 3 \cdot \frac{3}{7} \cdot \frac{3}{7} \cdot \frac{4}{7} = \frac{3.36}{343} = \frac{108}{343}
\]

\[
P(X = 3) = P(\text{RRR}) = \frac{3}{7} \cdot \frac{3}{7} \cdot \frac{3}{7} = \frac{27}{343}.
\]
The required probability distribution is

\[
\begin{array}{c|cccc}
 x & 0 & 1 & 2 & 3 \\
\hline
 P(X = x) & \frac{64}{343} & \frac{144}{343} & \frac{108}{343} & \frac{27}{343} \\
\end{array}
\]

Mean \quad = \quad E(X) \\
\quad = \quad \sum_{i} x_i p(x_i) \\
\quad = \quad 0 \left( \frac{64}{343} \right) + 1 \left( \frac{144}{343} \right) + 2 \left( \frac{108}{343} \right) + 3 \left( \frac{27}{343} \right) \\
\quad = \quad \frac{0 + 144 + 216 + 81}{343} \\
\quad = \quad \frac{441}{343} \\

E(X^2) \quad = \quad \sum_{i} x_i^2 p(x_i) \\
\quad = \quad 0^2 \left( \frac{64}{343} \right) + 1^2 \left( \frac{144}{343} \right) + 2^2 \left( \frac{108}{343} \right) + 3^2 \left( \frac{27}{343} \right) \\
\quad = \quad \frac{0 + 144 + 432 + 243}{343} \\
\quad = \quad \frac{819}{343} \\

Var(X) \quad = \quad E(X^2) - [E(X)]^2 \\
\quad = \quad \frac{819}{343} - \left( \frac{441}{343} \right)^2 \\
\quad = \quad \frac{819}{343} - \frac{194481}{117649} \\
\quad = \quad \frac{280917 - 194481}{117649} \\
\quad = \quad \frac{86436}{117649}
A random variable $X$ has the following distribution

<table>
<thead>
<tr>
<th>$X$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Find mean and variance.

**Solution:**

Mean = $E(X) = \sum x_i p(x_i)$

$$= -1 \left( \frac{1}{3} \right) + 0 \left( \frac{1}{6} \right) + 1 \left( \frac{1}{6} \right) + 2 \left( \frac{1}{3} \right)$$

$$= -\frac{1}{3} + 0 + \frac{1}{6} + \frac{2}{3} = \frac{0 + 1 + 4}{6} = \frac{3}{6} = \frac{1}{2}$$

$E(X^2) = \sum x_i^2 p(x_i)$

$$= (-1)^2 \left( \frac{1}{3} \right) + 0^2 \left( \frac{1}{6} \right) + 1^2 \left( \frac{1}{6} \right) + 2^2 \left( \frac{1}{3} \right)$$

$$= \frac{1}{3} + \frac{1}{6} + \frac{4}{3} = \frac{2 + 1 + 8}{6} = \frac{11}{6}$$

$Var(X) = E(X^2) - [E(X)]^2$

$$= \frac{11}{6} - \left( \frac{1}{2} \right)^2$$

$$= \frac{11}{6} - \frac{1}{4} = \frac{22 - 3}{12} = \frac{19}{12}$$
5.3 BINOMIAL DISTRIBUTION

Introduction:

Binomial distribution was discovered by James Bernoulli (1654-1705) in the year 1700 and was first published in 1713.

An experiment which has two mutually disjoint outcomes is called a Bernoulli trail. The two outcomes are usually called “success” and “failure”.

An experiment consisting of repeated number of Bernoulli trails is called Binomial experiment. A Binomial distribution can be used under the following conditions:

i. The number of trials is finite.
ii. The trials are independent of each other.
iii. The probability of success is constant for each trial.

Probability Function of Binomial Distribution

Let X denotes the number of success in n trial satisfying binomial distribution conditions. X is a random variable which can take the values 0,1,2,……,n, since we may get no success, one success,……or all n success.

The general expression for the probability of x success is given by

\[ P(X=x) = \binom{n}{x} p^x q^{n-x}, x = 0,1,2,3,\ldots,n. \]

where \( p \) = probability of success in each trial, \( q = 1-p \)

Definition: A random variable \( X \) is said to follow binomial distribution, if its probability mass function if given by

\[ P(X = x) = \binom{n}{x} p^x q^{n-x} : x = 0,1,2,3,\ldots,n. \]

\[ 0 \quad \text{Otherwise} \]

Where \( n, p \) are called parameters of the distribution.

Constants of the binomial distribution:

- Mean = np
- Variance = npq
- Standard Deviation = \( \sqrt{npq} \)
Note:

(i) \(0 \leq p \leq 1, 0 \leq q \leq 1\) and \(p + q = 1\)

(ii) In binomial distribution mean is always greater than variance.

(iii) To denote the random variable \(X\) which follows binomial distribution with parameters \(n\) and \(p\) is \(X \sim B(n, p)\).

WORKED EXAMPLES

PART - A

1. Comment, if any in the following statement: The mean of a binomial distribution is 5 and its standard deviation is 3.

Solution:

Given mean = 5
\[ np = 5 \quad \ldots 1 \]

Standard deviation = 3
\[ \sqrt{npq} = 3 \]

Squaring,
\[ npq = 9 \quad \ldots 2 \]

\[ \frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{9}{5} \]

\[ q = \frac{9}{5} > 1 \]

Hence, the given statement is not true.

2. Find \(n\) and \(p\) in the binomial distribution whose mean is 3 and variance is 2.

Solution:

Given, mean = 3
\[ np = 3 \quad \ldots 1 \]

Variance = 2
\[ npq = 2 \quad \ldots 2 \]

\[ \frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{2}{3} \quad (1) \]
\[ q = \frac{2}{3} \]
\[ \therefore p = 1 - q \]
\[ = 1 - \frac{2}{3} \]
\[ = \frac{1}{3} \]

Put \( p = \frac{1}{3} \) in \( n \) \( p = 3, \) \( n \times \frac{1}{3} = 3 \Rightarrow n = 9 \)

3. Find the mean of the binomial distribution if
\[ p(x) = 20c_x \left( \frac{2}{5} \right)^x \left( \frac{3}{5} \right)^{20-x} \]

Solution:
The binomial distribution is \( P(X=x) = nc_x p^x q^{n-x} \)

Here \( n = 20, \) \( p = \frac{2}{5}, \) \( q = \frac{3}{5} \)

Mean = \( np = 20 \times \frac{2}{5} = 8 \)

4. Write down the binomial distribution in which \( n = 8, \) \( p = \frac{3}{4} \)

Solution:

Here \( p = \frac{3}{4}, \) \( n = 8 \)
\[ q = 1 - p \]
\[ = 1 - \frac{3}{4} \]
\[ = \frac{1}{4} \]

The binomial distribution is \( P(X = x) = nc_x p^x q^{n-x} \)
\[ = 8c_x \left( \frac{3}{4} \right)^x \left( \frac{1}{4} \right)^{8-x} \]

Where \( x = 0, 1, 2, \ldots, 8 \)
5. In a binomial distribution if \( n=9 \) and \( p=\frac{1}{3} \), what is the value of variance.

Solution:

Given \( n = 9, p = \frac{1}{3} \)

\[ q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3} \]

Variance \[ = npq \]

\[ = 9 \cdot \frac{1}{3} \cdot \frac{2}{3} \]

\[ = 2 \]

6. A random variable \( X \) has the mean 6 and variance 2. If it is assumed that the distribution is binomial, find \( n \).

Solution:

Given mean = 6 variable = 2

\[ np = 6 \quad npq = 2 \]

\[ \frac{npq}{np} = \frac{2}{6} = \frac{1}{3} \quad \Rightarrow q = \frac{1}{3} \]

\[ p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3} \]

Mean \[ = np \]

\[ 6 = n \cdot \frac{2}{3} \]

\[ n = 9 \]

**PART - B**

1. In tossing 10 coins, what is the chance of having exactly 5 heads.

Solution:

Let \( X \) denote number of heads
\[ p = \text{probability of getting a head} = \frac{1}{2} \]
\[ q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2} \]
\[ n = \text{no of trials} = 10 \]

The binomial distribution is
\[ P(X = x) = \binom{n}{x} p^x q^{n-x} \]

P(getting exactly 5 heads) = \( P(X = 5) \)
\[ = 10 \binom{5}{5} \left( \frac{1}{2} \right)^5 \left( \frac{1}{2} \right)^{10-5} \]
\[ = \frac{63}{256} \]

2. A pair of dice is thrown 10 times, if getting a doublet is considered a success, find the probability of
   (i) 4 success (ii) no success

**Solution:**

Let \( X \) denote getting a doublet in a throw of a dice.

(A doublet means getting a pair is (1,1),(2,2),(3,3),(4,4),(5,5),(6,6)).

\[ p = P(\text{getting a doublet}) = \frac{6}{36} = \frac{1}{6} \]
\[ q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6} \]
\[ n = \text{number of trials} = 10 \]

(i) \( P(4 \text{ success}) = P(X = 4) \)
\[ = 10 \binom{4}{4} \left( \frac{1}{6} \right)^4 \left( \frac{5}{6} \right)^6 \]
\[ = \frac{210 \times 5^6}{6^{10}} = \frac{35 \left( \frac{5}{6} \right)^6}{216} \]
(ii) \( P(\text{no success}) = P(X = 0) = 10c_0 \left( \frac{1}{6} \right)^0 \left( \frac{5}{6} \right)^{10} = \left( \frac{5}{6} \right)^{10} \)

3. If the sum of mean and variance of a binomial distribution is 4.8 for 5 trials, find the distribution.

**Solution:**

Mean = \( np \)  
Variance = \( npq \)

Sum of mean and variance = 4.8

\[
np + npq = 4.8
\]

\[
np(1+q) = 4.8
\]

\[
5p(1+1-p) = 4.8 \quad (\therefore p + q = 1)
\]

\[
P^2 - 2p + 0.96 = 0 \implies p=1.2,0.8
\]

\[
p = 0.8 \quad q = 0.2 \quad (\therefore p \text{ cannot be greater than 1})
\]

The binomial distribution is \( p(X = x) = 5c_x (0.8)^x (0.2)^{5-x} \)

When \( x = 0, 1, 2, 3, 4, 5. \)

4. If on an average 1 ship out of 10 do not arrive safely to ports. Find the mean and the standard deviation of ships returning safely out of a total of 500 ships.

**Solution:**

Let \( X \) denote the ships arriving safely.

\[
p = P(\text{safe arrival}) = \frac{9}{10}
\]

\[
q = 1 - p = 1 - \frac{9}{10} = \frac{1}{10}
\]

\( n = 500 \)
mean = np = 500 × \( \frac{9}{10} \) = 450

variance = npq = 500 × \( \frac{9}{10} \) × \( \frac{1}{10} \) = 45

S.D = \( \sqrt{45} \) = 3\( \sqrt{5} \)

5. In a hurdle race a player has to cross 10 hurdles. The probability that he will clear each hurdle is \( \frac{5}{6} \). What is the probability that he will knock down less than 2 hurdles.

Solution:
Let X denote a player clearing the hurdle.

\[ q = \text{Probability of clearing} = \frac{5}{6} \]

\[ p = \text{probability of knocking} = 1 - \frac{5}{6} = \frac{1}{6} \]

\[ n = 10 \]

\[ P(\text{less than 2 hurdles}) = P(X < 2) = P(X = 0) + P(X = 1) = 10C_0 \left( \frac{1}{6} \right)^0 \left( \frac{5}{6} \right)^{10} + 10C_1 \left( \frac{1}{6} \right)^1 \left( \frac{5}{6} \right)^9 = \left( \frac{5}{6} \right)^{10} + \frac{10 \times 5^9}{6^{10}} \]

6. The overall percentage of passes in a certain examination is 80. If 6 candidates appear in the examination, what is the probability that at least 5 will pass the examination.

Solution:
Let X denote the overall percentage of passes.

\[ p = \frac{80}{100} = \frac{4}{5} \]

\[ q = 1 - p = 1 - \frac{4}{5} = \frac{1}{5} \]

\[ n = 6 \]
7. With usual notation find ‘p’ for binomial random variable if $n = 6$ and if $9P(X=4) = P(X=2)$.

Solution:

The binomial distribution is $P(X=x) = nc_x p^x q^{n-x}$

Given $9P(X=4) = P(X=2)$ and $n = 6$

$9 \times 6c_4 p^4 q^2 = 6c_2 p^2 q^4$

$9.15 p^4 q^2 = 15p^2 q^4$

$9 p^2 = q^2$

Taking square root

$3p = q$

$3p = 1-p$ \ (\therefore p + q = 1)

$4p = 1$

$p = \frac{1}{4}$

EXERCISE

PART - A

1. Define discrete random variable?

2. When a random variable is called a continuous random variable?

3. A random variable $X$ has the following distribution

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
P(X=x) 3a 4a 6a 7a 8a
find the value of a.

4. A discrete random variable takes values 0, 1, 2. Also if
P(X = 0) = \frac{144}{169}, P(X = 1) = \frac{1}{169} then find the value of if P(X=2)

5. If 3 coins are tossed simultaneously and X is the number of heads. What is the value of P(X=3).

6. Four coins are tossed at a time. If X denotes the number of tails, what are the possible values of X.

7. A random variable X has the following distribution function.

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X = x)</td>
<td>\frac{1}{6}</td>
<td>\frac{1}{3}</td>
<td>\frac{1}{3}</td>
<td>\frac{1}{6}</td>
</tr>
</tbody>
</table>

Find P(X≤2)

8. A random X has the following probability distribution function.

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X = x)</td>
<td>\frac{1}{8}</td>
<td>\frac{3}{8}</td>
<td>\frac{3}{8}</td>
<td>\frac{1}{8}</td>
</tr>
</tbody>
</table>

Find the distribution function F(x).

9. Examine whether f(x) = \frac{2x}{9}, 0 ≤ x ≤ 3 can be a pdf of a continuous random variable X?

10. The probability density function of a random value X is f(x) = Ax^2, 0 ≤ x ≤ 1, determine A.

11. If f(x) = kx^2, 0 ≤ x ≤ 3,

0 else where

is a pdf, find the value of k?
12. Verify whether \( f(x) = \frac{2}{\pi} \cdot \frac{1}{4 + x^2}, -\infty < x < \infty \) is a pdf?

13. If \( E(X) = 12 \) and \( E(X^2) = 200 \) what is \( \text{var}(X) \).

14. If \( E(X) = 3 \) and \( E(X^2) = 30 \), what is the variance of \( X \)

15. If \( \text{var}(X) = 2 \), what is \( \text{var}(5X + 7) \)?

16. If \( E(X) = 8 \) what is the value of \( E(3X) \).

17. A binomial distribution has mean 4 and variance \( \frac{8}{3} \) find \( p \) and \( n \).

18. A discrete random variable \( X \) has the mean 6 and variance 2. If it is assumed that the distribution is binomial find \( n \).

19. Find the S.D of the binomial distribution \( 10c_x \left( \frac{3}{5} \right)^x \left( \frac{2}{5} \right)^{10-x} \)

20. “For a binomial distribution mean is 9 and variance is 14”. Is it possible?

21. Find the mean of the binomial distribution \( 16c_x \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{16-x} \)

PART - B

1. A random variable \( X \) has the following probability distribution function.

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>3a</td>
<td>5a</td>
<td>7a</td>
<td>9a</td>
<td>11a</td>
<td></td>
</tr>
</tbody>
</table>

Find

(i) The value of \( a \)

(ii) \( P(X < 4) \)

(iii) \( P(X \geq 3) \)

(iv) \( P(2 < X < 5) \)
2. A random variable $X$ has the following probability distribution

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$k$</td>
<td>$3k$</td>
<td>$5k$</td>
<td>$7k$</td>
<td>$9k$</td>
<td>$11k$</td>
<td>$13k$</td>
<td>$15k$</td>
<td>$17k$</td>
</tr>
</tbody>
</table>

i. Determine the value of 'k'

ii. $P(X < 5)$

iii. $P(X \geq 4)$

iv. $P(0 \leq X < 5)$

3. The probability density function of a random variable $X$ is

$$f(x) = \begin{cases} 
3x^2, & 0 \leq x \leq 1 \\
0, & \text{elsewhere}
\end{cases}$$

If (i) $P(X \leq a) = P(X > a)$ and (ii) $P(X > b) = 0.05$

Find the value of $a$ & $b$

4. If $f(x) = \begin{cases} 
\frac{A}{x}, & 1 < x < e^3 \\
0, & \text{otherwise}
\end{cases}$

is the pdf of a random variable $X$, find $p$

$P(X > e)$.

5. The amount of bread (in hundred of kg) $x$, that a certain bakery is able to sell in a month is found to be a numerical valued random phenomenon specified by probability density function $f(x)$ given by

$$f(x) = \begin{cases} 
Ax, & 0 \leq x \leq 5 \\
A(10 - x), & 5 \leq x \leq 10 \\
0, & \text{elsewhere}
\end{cases}$$

i. Find the value of $A$

ii. What is the probability that the number of kg of bread that will be sold next month is,

(a) more than 500 kg

(b) Between 250 and 750 kg?
6. Two cards are drawn from a well shuffled pack of 52 cards with replacement, find the mean and variance of the number of aces.

7. The probability that a student will graduate is 0.4. Find the probability that out of 5 students (i) none (ii) one (iii) at least one will be a graduate.

8. A player tosses 3 fair coins. He wins Rs.5 if 3 heads appear, Rs.3 if 2 heads appear, Rs.1 if 1 head occurs. On the other head he loses Rs.15/- if all tails occur. Find the expected gain.

9. If a random variable $X$ has the following probability distribution

<table>
<thead>
<tr>
<th>$X$</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{6}$</td>
<td>$\frac{1}{3}$</td>
</tr>
</tbody>
</table>

Find (i) $E(X)$ (ii) $\text{var}(X)$ (iii) $E(2X + 3)^2$

10. A game is played with a single fair die. A player wins Rs.20 if a 2 turns up, Rs 40 of a 4 turns up, loses Rs.30 if a 6 turns up. While he neither wins nor loses if any other face turns up. Find the expected sum of money he can win.

11. Ten coins are tossed simultaneously. Find the probability of getting exactly 2 heads.

12. In a binomial distribution having 6 independent trials the probabilities of 0 and 1 successes are 0.4 and 0.2 respectively. Find $p$ and $P(X=0)$.

13. With usual notation find ‘$p$’ for the binomial distribution $X$ if $n=6$ and if $9P(X =4)=P(X =2)$

14. Find the probability that in a family of 4 children there will be at least 1 boy and 1 girl.

15. Ten coins are tossed simultaneously. Find the probability of getting (i) at least seven heads (ii) exactly 7 heads and (iii) almost seven heads.

16. In a binomial distribution with 5 independent trials the probability of getting 1 and 2 success are 0.6 and 0.2 respectively. Find $p$. 

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17. In a large consignment of iron boxes 10% are defective. A random sample of 20 is taken for inspection. Find the probability that at the most there are 3 defective iron boxes.

18. The mean and variance of a binomial variate X with parameters n and p are 16 and 8 respectively. Find P(X =0) and P(X =1)

19. Four coins are tossed simultaneously probability distribution. Find the probability of getting at least 2 heads.

20. A random variable X has the following probability distribution

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X = x)</td>
<td>a</td>
<td>4a</td>
<td>6a</td>
<td>7a</td>
<td>8a</td>
</tr>
</tbody>
</table>

Find the value of a and find E(X^2 + X)

ANSWERS

PART-A

3. \( a = \frac{1}{28} \)

4. \( \frac{24}{169} \)

5. \( \frac{1}{8} \)

6. 0, 1, 2, 3, 4

7. \( \frac{5}{6} \)

8. \( \begin{array}{c|cccc}
     x & 0 & 1 & 2 & 3 \\
     F(x) & \frac{1}{8} & \frac{4}{8} & \frac{7}{8} & 1 \\
  \end{array} \)

10. \( A = 3 \)

11. \( k = \frac{1}{9} \)

13. 56

14. 21

15. 50

16. 24

17. \( p = \frac{1}{3} \)

18. 9

19. \( \sqrt{\frac{12}{5}} \)

20. not possible

21. 8
PART - B

1. (i) \( a = \frac{1}{36} \) (ii) \( \frac{4}{9} \) (iii) \( \frac{3}{4} \) (iv) \( \frac{4}{9} \)

2. (i) \( a = \frac{1}{81} \) (ii) \( \frac{25}{81} \) (iii) \( \frac{65}{81} \) (iv) \( \frac{25}{81} \)

3. \( a = \left( \frac{1}{2} \right)^3 \) \( b = \left( \frac{19}{20} \right)^3 \)

4. (i) \( A = \frac{1}{3} \) (ii) \( \frac{2}{3} \)

5. (i) \( A = \frac{1}{25} \) (ii) (a) \( \frac{1}{2} \) (b) 0.75

6. \( \frac{2}{13}, \frac{24}{169} \)

7. (i) 0.0776 (ii) 0.2592 (iii) 0.9224

8. Rs.0.25

9. (i) \( \frac{1}{2} \) (ii) \( \frac{14}{12} \) (iii) \( \frac{67}{3} \)

10. Rs.5

11. \( \frac{45}{2^{10}} \)

12. (i) \( \frac{1}{13} \) (ii) \( \left( \frac{12}{13} \right)^6 \)

13. \( \frac{1}{4} \)

14. \( \frac{7}{8} \)

15. (i) \( \frac{11}{64} \) (ii) \( \frac{15}{128} \) (iii) \( \frac{121}{125} \)

16. \( p = \frac{1}{7} \)

17. 0.8666

18. \( n = 32 \) \( p = \frac{1}{2} \) (ii) \( \frac{1}{2^{32}} \) (iii) \( \frac{1}{2^{27}} \)

19. \( \frac{11}{16} \)

20. (i) \( a = \frac{1}{28} \) (ii) \( \frac{72}{7} \)